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Abstract

In classical continuum mechanics, quasi-linear systems of conservation laws can be symmetrized if they admit an additional convex conservation law. In particular, this implies the hyperbolicity of governing equations. For capillary fluids, the internal energy depends not only on the density but also on its derivatives with respect to space variables. Consequently, the governing equations belong to the class of dispersive systems. In that case we propose a symmetric form of governing equations which is different from the classical Godunov - Friedrichs - Lax representation. This new symmetric form implies the stability of constant solutions.

1 Introduction

Quasi-linear systems of conservation laws can be symmetrized, if they admit an additional *convex* conservation law (Godunov, 1961, Friedrichs and Lax, 1971). The symmetric form implies hyperbolicity of governing equations. For conservation laws with vanishing right-hand side, the hyperbolicity is equivalent to stability of constant solutions with respect to perturbations of the form $e^{i(\mathbf{k}^*\mathbf{x}-\lambda t)}$, $i^2=-1$, $\mathbf{k}^*=(k_1,\cdots,k_n)$, $\mathbf{x}=(x^1,\cdots,x^n)$, where "$*$" denotes the transposition. Indeed, the following symmetric form of governing equations for an unknown vector variable $\mathbf{v}$

$$A\frac{\partial \mathbf{v}}{\partial t}+\sum_{i=1}^{n}B^i\frac{\partial \mathbf{v}}{\partial x^i}=0 \tag{1}$$

where matrix $A=A^*$ is positive definite, $B^i=(B^i)^*$, implies the dispersion relation

$$\mathtt{det}(B-\lambda A)=0,\quad B=\sum_{i=1}^{n}B^i k_i \tag{2}$$

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which determines real values of λ for any real wave vector $\mathbf{k}$.

In this note we get an analog of symmetric form (1) for equations of *capillary fluids* that belong to the class of *dispersive systems*, because the internal energy depends not only on the density but also on its derivatives with respect to space variables. We will see that the analog of equation (2) is

$$\det(B + iC - \lambda A) = 0 \tag{3}$$

where $C = -C^*$ is an antisymmetric matrix depending on the wave vector $\mathbf{k}$. Since $B + iC$ is Hermitian matrix and the symmetric matrix A is positive definite, all the frequencies λ are also real. For a capillary fluid the matrix C is of the form

$$C = -C^* = \begin{pmatrix} 0 & \mathbf{0}^* & \mathbf{0}^* \\ \mathbf{0} & O & -\rho_e \mathbf{k}\mathbf{k}^* \\ \mathbf{0} & \rho_e \mathbf{k}\mathbf{k}^* & O \end{pmatrix}$$

where ρ_e is the equilibrium fluid density and O is the zero-matrix 3×3.

Here and later, for any vectors $\mathbf{a}, \mathbf{b}$ we use the notation $\mathbf{a}^*\mathbf{b}$ for the scalar product (the line is multiplied by the column vector) and $\mathbf{a}\ \mathbf{b}^*$ for the tensor product (or $\mathbf{a}\otimes\mathbf{b}$ the column vector is multiplied by the line vector). Divergence of a linear transformation A is the covector $\mathrm{div}(A)$ such that, for any constant vector $\mathbf{a}$, $\mathrm{div}(A)\ \mathbf{a} = \mathrm{div}(A\mathbf{a})$. The identical transformation is denoted by I.

In section 2 we present the multi-dimensional case in Eulerian coordinates for a particular form of the internal energy. In section 3 we consider in Lagrangian coordinates a one-dimensional case for the general form of internal energy.

2 Governing equations in Eulerian coordinates

The internal energy per unit volume of a capillary fluid is taken in the form

$$e(\rho, \eta, \mathbf{w}) = \varepsilon(\rho, \eta) + \frac{c\,|\mathbf{w}|^2}{2} \tag{4}$$

where ρ is the fluid density, $\mathbf{w} = \mathrm{grad}\,\rho$ (or $\mathbf{w}^* = \nabla\rho$), η is the entropy per unit volume, c is the *capillarity coefficient* which is assumed to be a constant [Rocard, 1952, Rowlinson and Widom, 1984]. The *homogeneous* energy ε satisfies the Gibbs identity

$$d\varepsilon = \mu\ d\rho + \theta\ d\eta \tag{5}$$

where $\mu = (\varepsilon + P - \theta\eta)/\rho$ is the chemical potential, θ is the temperature, $P = \rho\dfrac{\partial\varepsilon}{\partial\rho} - \varepsilon$ is the thermodynamic pressure. By using Hamilton's principle, governing equations of such a fluid were obtained by Casal (1972) (see also Casal and Gouin, 1985, Gavrilyuk and Shugrin, 1996). They are in the form

$$\begin{gathered} \rho_t + \mathrm{div}\ \mathbf{j} = 0 \\ \eta_t + \mathrm{div}\left(\frac{\eta}{\rho}\,\mathbf{j}\right) = 0 \\ \mathbf{j}^*_t + \mathrm{div}\left(\frac{\mathbf{j}\mathbf{j}^*}{\rho} + p\,I + c\mathbf{w}\mathbf{w}^*\right) = \mathbf{0}^* \end{gathered} \tag{6}$$

where $\mathbf{j} = \rho\mathbf{u}$, $\mathbf{u}$ is the velocity vector, index t is the partial derivative with respect to time, $p = \rho\dfrac{\delta\varepsilon}{\delta\rho} - e = P - c\left(\rho\,\mathrm{div}\,\mathbf{w} + \dfrac{|\mathbf{w}|^2}{2}\right)$, where $\dfrac{\delta}{\delta\rho}$ means the variational derivative with respect to ρ. By using (4) and (5) we can obtain from (6) the energy conservation law

$$\left(\varepsilon + \frac{|\mathbf{j}|^2}{2\rho} + \frac{c\,|\mathbf{w}|^2}{2}\right)_t + \mathrm{div}\left(\left(\varepsilon + \frac{|\mathbf{j}|^2}{2\rho} + P\right)\mathbf{u} + c\,(\mathbf{w}\,\mathrm{div}\;\mathbf{j} - \mathbf{j}\;\mathrm{div}\;\mathbf{w})\right) = 0 \tag{7}$$

Since

$$\begin{aligned}
&\mathrm{div}\left(c\mathbf{w}\mathbf{w}^* - c\left(\rho\,\mathrm{div}\;\mathbf{w} + \frac{c\,|\mathbf{w}|^2}{2}I\right)\right)\\
&= c\mathbf{w}^*\,\mathrm{div}\;\mathbf{w} + c\mathbf{w}^*\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)^* - c\mathbf{w}^*\,\mathrm{div}\;\mathbf{w} - c\rho\boldsymbol{\nabla}\,(\mathrm{div}\;\mathbf{w}) - c\mathbf{w}^*\left(\frac{\partial\mathbf{w}}{\partial\mathbf{x}}\right)\\
&= -c\rho\boldsymbol{\nabla}\,(\mathrm{div}\;\mathbf{w})
\end{aligned}$$

the momentum equation reads

$$\mathbf{j}^*_t + \mathrm{div}\left(\frac{\mathbf{j}\mathbf{j}^*}{\rho} + P\,I\right) - c\rho\boldsymbol{\nabla}\,(\mathrm{div}\;\mathbf{w}) = \mathbf{0}^*$$

The gradient of the mass conservation law verifies another conservation law

$$\mathbf{w}^*_t + \boldsymbol{\nabla}\,(\mathrm{div}\;\mathbf{j}) = 0$$

If we add an initial condition such that

$$\mathbf{w}^*\big|_{\,t=0} = \boldsymbol{\nabla}\rho\big|_{\,t=0}$$

we can consider $\mathbf{w}$ as an independent variable. The fact that $\mathbf{w}^* = \boldsymbol{\nabla}\rho$ will be a consequence of the governing equations. Finally, we obtain equations (6) in the following equivalent non-divergence form

$$\begin{gathered}
\rho_t + \mathrm{div}\;\mathbf{j} = 0\\
\eta_t + \mathrm{div}\,\frac{\eta}{\rho}\,\mathbf{j} = 0\\
\mathbf{j}^*_t + \mathrm{div}\left(\frac{\mathbf{j}\mathbf{j}^*}{\rho} + P\,I\right) - c\rho\boldsymbol{\nabla}\,(\mathrm{div}\;\mathbf{w}) = \mathbf{0}^*\\
\mathbf{w}^*_t + \boldsymbol{\nabla}\,(\mathrm{div}\;\mathbf{j}) = 0
\end{gathered} \tag{8}$$

The theory of capillary fluids is usually applied for van der Waals-like fluids. For such fluids the energy $\varepsilon\,(\rho,\eta)$ is not convex for all values of ρ and η. We suppose that we are in the vicinity of an equilibrium state (ρ_e,η_e) where the energy function is locally convex. Let us introduce conjugate variables $(q,\ \theta,\ \mathbf{u},\ \mathbf{r})$ by the formula

$$\begin{aligned}
dE \equiv d\left(\varepsilon + \frac{|\mathbf{j}|^2}{2\rho} + c\frac{|\mathbf{w}|^2}{2}\right) &\equiv \left(\mu - \frac{|\mathbf{u}|^2}{2}\right)d\rho + \theta d\eta + \mathbf{u}^* d\mathbf{j} + c\;\mathbf{w}^* d\mathbf{w}\\
&\equiv q d\rho + \theta d\eta + \mathbf{u}^* d\mathbf{j} + \mathbf{r}^* d\mathbf{w}
\end{aligned} \tag{9}$$

The Lagrange transformation of the total energy E is defined by

$$\Pi = \rho q + \eta\theta + \mathbf{j}^*\mathbf{u} + \mathbf{w}^*\mathbf{r} - E = P + \frac{|\mathbf{r}|^2}{2c}$$

where the thermodynamic pressure P is considered as a function of $q,\ \theta$ and $\mathbf{u}$. Hence, in terms of the conjugate variables $(q,\ \theta,\ \mathbf{u},\ \mathbf{r})$ defined by equation (9), equations (8) can be rewritten in the following form

$$\begin{gathered}
\left(\frac{\partial \Pi}{\partial q}\right)_t + \operatorname{div}\left(\frac{\partial(\Pi\mathbf{u})}{\partial q}\right) = 0 \\
\left(\frac{\partial \Pi}{\partial \theta}\right)_t + \operatorname{div}\left(\frac{\partial(\Pi\mathbf{u})}{\partial \theta}\right) = 0 \\
\left(\frac{\partial \Pi}{\partial \mathbf{u}}\right)_t + \operatorname{div}\left(\frac{\partial(\Pi\mathbf{u})}{\partial \mathbf{u}} - \frac{\partial \Pi}{\partial q}\frac{\partial \mathbf{r}}{\partial \mathbf{x}}\right) = 0 \\
\left(\frac{\partial \Pi}{\partial \mathbf{r}}\right)_t + \operatorname{div}\left(\frac{\partial(\Pi\mathbf{u})}{\partial \mathbf{r}} + \frac{\partial \Pi}{\partial q}\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) = 0
\end{gathered} \tag{10}$$

If the capillary coefficient c is zero, $\Pi = P$ and we get the gas dynamics equation and the symmetric form of Godunov (1961).

Multiplying equations (10) by $q,\ \theta,\ \mathbf{u}$ and $\mathbf{r}$, summing up all of them and using the identity

$$\operatorname{rot}\ (\mathbf{a}\times\mathbf{b}) = \ [\mathbf{a},\mathbf{b}] + \mathbf{a}\operatorname{div}\mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{a}$$

where

$$[\mathbf{a},\mathbf{b}] = \frac{\partial \mathbf{a}}{\partial \mathbf{x}}\mathbf{b} - \frac{\partial \mathbf{b}}{\partial \mathbf{x}}\mathbf{a}$$

denotes the Poisson bracket, we get the conservation of the energy (7) in the form

$$\begin{gathered}
\left(q\frac{\partial \Pi}{\partial q} + \theta\frac{\partial \Pi}{\partial \theta} + \frac{\partial \Pi}{\partial \mathbf{u}}\mathbf{u} + \frac{\partial \Pi}{\partial \mathbf{r}}\mathbf{r} - \Pi\right)_t \\
+ \operatorname{div}\left(q\frac{\partial(\Pi\mathbf{u})}{\partial q} + \theta\frac{\partial(\Pi\mathbf{u})}{\partial \theta} + \frac{\partial(\Pi\mathbf{u})}{\partial \mathbf{u}}\mathbf{u} + \frac{\partial(\Pi\mathbf{u})}{\partial \mathbf{r}}\mathbf{r} - \Pi\mathbf{u}\right. \\
\left. + \frac{\partial \Pi}{\partial q}\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{r} - \frac{\partial \Pi}{\partial q}\frac{\partial \mathbf{r}}{\partial \mathbf{x}}\mathbf{u}\right) = \mathbf{0}
\end{gathered}$$

The system (10) admits constant solutions $(\rho_e, \eta_e, \mathbf{u}_e, \mathbf{w}_e = \mathbf{0})$. Since the governing equations are invariant under Galilean transformation, we can assume that $\mathbf{u}_e = \mathbf{0}$. If we look for the solution of the linearized system proportional to $e^{i(\mathbf{k}^*\mathbf{x}-\lambda t)}$, we get equation (3), in which we have put

$$A = \frac{\partial}{\partial \mathbf{v}}\left(\left(\frac{\partial \Pi}{\partial \mathbf{v}}\right)^*\right), \qquad B = \sum_{i=1}^{n} B^i k_i, \qquad B^i = \frac{\partial}{\partial \mathbf{v}}\left(\left(\frac{\partial \Pi u^i}{\partial \mathbf{v}}\right)^*\right),$$

$$C = -C^* = \begin{pmatrix} 0 & \mathbf{0}^* & \mathbf{0}^* \\ \mathbf{0} & O & -\rho_e\mathbf{k}\mathbf{k}^* \\ \mathbf{0} & \rho_e\mathbf{k}\mathbf{k}^* & O \end{pmatrix} \quad \text{with} \quad O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\mathbf{v}^* = (q,\ \theta,\ \mathbf{u}^*,\ \mathbf{r}^*)$. Hence, eigenvalues λ are real if A is positive definite.

3 One-dimensional barotropic case

In mass Lagrangian coordinates (t, z) the governing equations are (see Gavrilyuk and Serre, 1995)

$$v_t - u_z = 0, \quad u_t + p_z = 0, \quad p = -\frac{\delta e}{\delta v} = -\left(\frac{\partial e}{\partial v} - \frac{\partial}{\partial z}\left(\frac{\partial e}{\partial v_z}\right)\right), \quad e = e(v, v_z)$$

where $v = \dfrac{1}{\rho}$ denotes the specific volume. This case is general: we do not suppose a particular form (4) of the energy e. Consider an augmented system

$$\begin{gathered} v_t - u_z = 0 \\ w_t - u_{zz} = 0 \\ u_t - \left(\frac{\partial e}{\partial v} - \frac{\partial}{\partial z}\left(\frac{\partial e}{\partial w}\right)_z\right)_z = 0 \end{gathered} \tag{11}$$

Let us define π and the conjugate variables (σ, r) as

$$\pi = \frac{\partial e}{\partial v}\, v + \frac{\partial e}{\partial w}\, w \equiv \sigma v + r w - e$$

In terms of π and (σ, r, u) the system (11) reads

$$\begin{gathered} \left(\frac{\partial \pi}{\partial \sigma}\right)_t - u_z = 0 \\ \left(\frac{\partial \pi}{\partial r}\right)_t - u_{zz} = 0 \\ u_t - (\sigma - r_z)_z = 0 \end{gathered}$$

In matrix form we get

$$A \begin{pmatrix} \sigma \\ r \\ u \end{pmatrix}_t + B_1 \begin{pmatrix} \sigma \\ r \\ u \end{pmatrix}_z + C_1 \begin{pmatrix} \sigma \\ r \\ u \end{pmatrix}_{zz} = 0 \tag{12}$$

where

$$A = \begin{pmatrix} \dfrac{\partial^2 \pi}{\partial \sigma^2} & \dfrac{\partial^2 \pi}{\partial \sigma \partial r} & 0 \\ \dfrac{\partial^2 \pi}{\partial \sigma \partial r} & \dfrac{\partial^2 \pi}{\partial r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; B_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{13}$$

Equations (12) and (13) imply a dispersion relation of type (3), if we put $B = kB_1$, $C = k^2C_1$. We note also that the system admits the energy conservation law

$$\Big(\frac{u^2}{2} - \sigma\frac{\partial\pi}{\partial\sigma} + r\frac{\partial\pi}{\partial r} - \pi\Big)_t + \Big(-\sigma u + [r,u]\Big)_z = 0$$

where $[r,u] = ur_z - ru_z$.

Remark. Analogous symmetric forms may be obtained for bubbly liquids, where the internal energy is a function not only of the density but also of the total derivative of the density with respect to time.

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